An Extension of Operational Calculus

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THE Heaviside operational calculus postulates at the outset that the initial (boundary) conditions at reference time $t = 0$ are those of equilibrium; that is to say, the system is at rest when suddenly energized at time $t = 0$ by a "unit" impressed force. By unit impressed force is to be understood a force which is zero before, unity after, time $t = 0$.

In a paper published in Volume 7, 1929, of the *Philosophical Magazine*, Van der Pol briefly indicated the appropriate procedure for extending the operational calculus to cover arbitrary initial conditions. The present paper is an exposition of this generalization for a system of a finite number of degrees of freedom, followed by an application to the differential equations of the transmission line. While stated in the language of electric circuit theory, it is to be understood that the processes are generally applicable to a wide variety of problems.

We start with the canonical equations for a network of n degrees of freedom

$$\begin{aligned} z_{11}I_1 + z_{12}I_2 + \cdots + z_{1n}I_n &= E_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ z_{n1}I_1 + z_{n2}I_2 + \cdots + z_{nn}I_n &= E_n \end{aligned} \tag{1}$$

where

$$z_{jk} = \left(L_{jk}\frac{d}{dt} + R_{jk} + \frac{1}{C_{jk}}\int_{-\infty}^{t} dt \right). \tag{2}$$

Now multiply the equations (1) by e^{-pt} throughout and integrate from 0 to infinity. Also let J_m and F_m denote the Laplace transforms of I_m and E_m; thus

$$\begin{aligned} J_m &= \int_0^\infty I_m e^{-pt}\,dt, \\ F_m &= \int_0^\infty E_m e^{-pt}\,dt. \end{aligned} \tag{3}$$

Now let I_m^0 and Q_m^0 denote the initial values (at time $t = 0$) of I_m and the charge Q_m in the mth mesh; also let us replace z_{jk} of (2) hy

$$z_{jk} = pL_{jk} + R_{jk} + 1/pC_{jk}. \tag{4}$$

We then have, replacing (1), the system of *algebraic* equations:

$$\begin{aligned} z_{11}J_1 + z_{12}J_2 + \cdots + z_{1n}J_n &= F_1 + G_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots &\cdots\cdots, \\ z_{n1}J_1 + z_{n2}J_2 + \cdots + z_{nn}J_n &= F_n + G_n. \end{aligned} \tag{5}$$

In these equations, $G_1, G_2, \cdots, G_n$ denote the following summations:

$$\begin{aligned} G_1 &= L_{11}I_1{}^0 - Q_1{}^0/C_{11}p + L_{12}I_2{}^0 - Q_2{}^0/C_{12}p + \cdots \\ &\qquad\qquad + L_{1n}I_n{}^0 - Q_n{}^0/C_{1n}p, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ G_n &= L_{n1}I_1{}^0 - Q_1{}^0/C_{n1}p + \cdots + L_{nn}I_n{}^0 - Q_n{}^0/C_{nn}p. \end{aligned} \tag{6}$$

The right hand sides of equations (5) are thus known in terms of the impressed forces and the specified initial values of the currents and charges. They can therefore be solved in the usual manner for $J_1, \cdots, J_n$. Thus

$$J_m = \frac{F_1 + G_1}{Z_{m1}} + \frac{F_2 + G_2}{Z_{m2}} + \cdots + \frac{F_n + G_n}{Z_{mn}}. \tag{7}$$

Having thus determined $J_1, \cdots, J_n$ as functions of p, $I_1, \cdots, I_n$ are determined as functions of time by the Laplace integral equation:

$$J_m(p) = \int_0^\infty I_m(t)e^{-pt}\,dt, \qquad p_R > c, \tag{8}$$

which completes the formal solution of the problem. Note that if $G_1 = G_2 = \cdots = G_n = 0$, the solution reduces to the usual form.

Equations (6) for $G_1, \cdots, G_n$, may be written in a compact and elegant form as follows: Let

$$\begin{aligned} T &= \tfrac{1}{2}\sum\sum L_{jk}I_jI_k, \\ U &= \tfrac{1}{2}\sum\sum \frac{1}{C_{jk}} Q_jQ_k. \end{aligned} \tag{9}$$

T is then the kinetic or magnetic energy stored in the network and U is the corresponding potential or electric energy. Then

$$G_m = \left(\frac{\partial T}{\partial I_m}\right)_{t=0} - \frac{1}{p}\left(\frac{\partial U}{\partial Q_m}\right)_{t=0}, \; m = 1, 2, \cdots, n. \tag{10}$$

The foregoing solution is compact, elegant and formally complete. In practical applications to networks of many degrees of freedom it may well present formidable difficulties in computation and interpre-

tation. This, however, is merely answerable to the complexity of the physical problem, and no simpler general solution can possibly exist.

The foregoing method when applied to the differential equations of the transmission line, leads to the following differential equations

$$
\begin{aligned}
(Lp + R)J &= -\frac{\partial}{\partial x}\Phi + LI^0, \\
(Cp + G)\Phi &= -\frac{\partial}{\partial x}J + CV^0.
\end{aligned}
\tag{11}
$$

Here J and Φ are Laplace transforms of the current I and voltage V and I^0, V^0 are the initial values of I and V at reference time $t = 0$. J, Φ, I^0, V^0 are functions of x but of course independent of t.

The formal solution of equations (11) is as follows: write

$$
\begin{aligned}
Lp + R &= Z(p) = Z, \\
Cp + G &= Y(p) = Y, \\
\sqrt{ZY} &= \gamma, \qquad \sqrt{Z/Y} = K.
\end{aligned}
\tag{12}
$$

Also

$$LI^0 - \frac{C}{Y}\frac{\partial}{\partial x}V^0 = F(x) = F.$$

Then

$$
\begin{aligned}
J = {} & e^{-\gamma x}\left\{ A + \frac{1}{2K}\int^x dyF(y)e^{\gamma y} \right\} \\
& - e^{\gamma x}\left\{ B + \frac{1}{2K}\int^x dyF(y)e^{-\gamma y} \right\},
\end{aligned}
\tag{13}
$$

$$\Phi = -\frac{K}{\gamma}\frac{\partial}{\partial x}J + \frac{C}{Y}V^0. \tag{14}$$

A and B are constants of integration determined by the relations between J and Φ at the physical terminals of the line.